Technical Notes

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Comparison of the Convergence Characteristics of Two Conformal Mapping Methods

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Introduction

CONFORMAL mapping has been used for many years for the potential-flow analysis of single-element airfoils. In recent years its use has been extended to include the analysis of two-element airfoils by Ives[1] and general multielement airfoils by Halsey.[2] Each of these new multiple-body methods makes use of an older single-body method at some stage in the calculations; Ives' uses Theodorsen's[3] method, and Halsey's uses James'[4] method. It is well known that the iterations in Theodorsen's method are convergent only if the body being transformed is sufficiently close to circular. This limitation does not apply to James' method, and, hence, James' method is more suitable for use in cases with complicated body geometry. This paper compares the procedures used by James' and Theodorsen's methods and provides an explanation for the superior convergence characteristics of the iterations in James' method.

General Description of Single-Body Mapping Methods

The mathematical problem under consideration is the conformal transformation of the region outside a simple, smooth, closed Jordan curve to the region outside a unit circle. More general curves (curves with corners or curves that do not close) could be considered, either by the use of preliminary transformations or by the inclusion of singular terms in the series for the mapping function, but the essence of the following arguments would be unchanged. The further assumption is made that the curve (or body) being transformed is of the proper scale and orientation so that it can be mapped to a unit circle while distant regions are left undisturbed. Since the transformations can be performed without prior knowledge of the scale factor or rotation angle required to meet this condition, this assumption also leaves the essence of the following arguments unchanged.

Under the above conditions, the transformation can be represented by a function having the form

$$z=\zeta+a_0+a_1/\zeta+a_2/\zeta^2+\ldots \tag{1}$$

where z and ζ are the complex coordinates in the physical and circle planes, respectively, and $a_0,a_1,a_2,\ldots$ are complex coefficients. Neither James' nor Theodorsen's method makes use of this series directly, but each uses a closely related series.

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The two methods are remarkably similar, and it is possible to refer to each as the "method of successive conjugates." The following subsections describe these two methods in more detail.

Theodorsen's Mapping Method

Theodorsen's method makes use of a series of the following form:

$$\log(z/\zeta)=b_0+b_1/\zeta+b_2/\zeta^2+\ldots \tag{2}$$

Truncating the series after N terms and applying it at equally spaced points on the circle (going clockwise around the perimeter) gives

$$\log(z/\zeta)_j=\sum_{k=0}^{N-1} b_k e^{+i2\pi jk/N} \tag{3}$$

The term on the left-hand side can be related to the geometric variables as follows:

$$\log(z/\zeta)_j=\log r_j+i(\theta-\omega)_j \tag{4}$$

where r_j is the radial coordinate of point j in the z plane and θ_j and ω_j are the angular coordinates (positive clockwise) of the points in the z and ζ planes, respectively. (These geometric quantities are illustrated in Fig. 1.) The real and imaginary parts of $\log(z/\zeta)$ are conjugate harmonic functions; given one part, the other can be found in an efficient manner using fast Fourier transforms. These relationships make possible the following iteration procedure:

1) Calculate the values of $\log r$ and θ at the defining points of the body in the z plane and determine the curve-fit coefficients of $\log r$ vs θ.

2) Estimate the values of θ_j of the points in the z plane corresponding to the equally spaced points in the ζ plane. ($\theta_j=\omega_j$ is often assumed.)

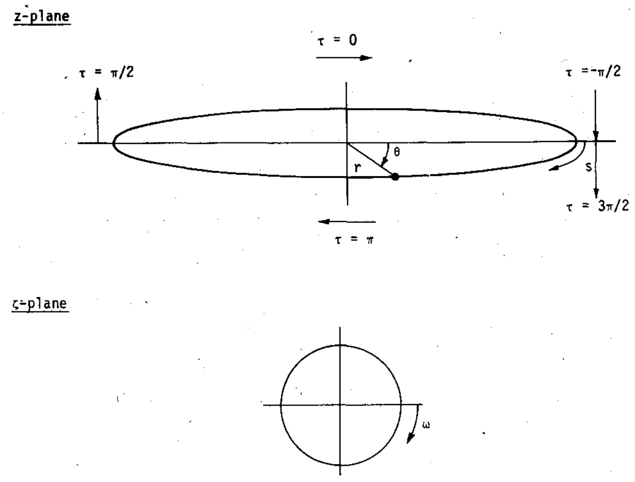

Fig. 1 Definition of the geometric quantities used in the mapping calculations.

3) Use the curve-fit coefficients to determine the values of $\log r_j$ corresponding to the estimated values of θ_j.

4) Perform a conjugate function calculation to determine values of $(\theta-\omega)_j$ corresponding to the most recent values of $\log r_j$, and use these to update the estimated values of θ_j.

5) Iterate steps 3 and 4 until the values of θ_j converge.

6) Determine the coefficients of the mapping function from the converged results.

James' Mapping Method

James' method makes use of a series of the following form:

$$\log\left(\frac{dz}{d\zeta}\right) = c_0 + c_1/\zeta + c_2/\zeta^2 + \ldots \tag{5}$$

Truncating the series after N terms and applying it at equally spaced points on the circle gives

$$\log\left(\frac{dz}{d\zeta}\right)_j = \sum_{k=0}^{N-1} c_k e^{+i2\pi jk/N} \tag{6}$$

The term on the left-hand side is related to the geometric variables as follows:

$$\log\left(\frac{dz}{d\zeta}\right)_j = \log\left|\frac{dz}{d\zeta}\right|_j + i\arg\left(\frac{dz}{d\zeta}\right)_j \tag{7}$$

$$s_j = \int_0^{\omega_j}\left|\frac{dz}{d\zeta}\right| d\omega \tag{8}$$

$$\arg\left(\frac{dz}{d\zeta}\right)_j = \tau_j + \omega_j - 3\pi/2 \tag{9}$$

where s_j is the surface arc length in the physical plane and τ_j is the surface angle (according to the convention illustrated in Fig. 1). These relationships make possible the following iteration procedure:

1) Calculate the values of τ and s at the defining points of the body in the z plane and determine the curve-fit coefficients of τ vs s.

2) Estimate the values of $|dz/d\zeta|_j$ at equally spaced points on the circle. ($|dz/d\zeta|_j = 1.0$ is usually assumed.)

3) Perform the integration of Eq. (8) to obtain estimates of the values of s_j.

4) Use the curve-fit coefficients to determine values of τ_j corresponding to the estimated values of s_j, and calculate $\arg(dz/d\zeta)_j$ using Eq. (9).

5) Perform a conjugate function calculation to determine values of $\log|dz/d\zeta|_j$ corresponding to the most recent values of $\arg(dz/d\zeta)_j$ and take the exponential to update the estimated values of $|dz/d\zeta|_j$.

6) Iterate steps 3-5 until the values of $|dz/d\zeta|_j$ converge.

7) Determine the coefficients of the mapping function from the converged results.

Comparison of the Convergence Properties of the Two Methods

A look at the physical data for which curve fits must be obtained in the two methods immediately identifies a broad class of shapes for which James' method is more suitable than Theodorsen's. For only moderately complicated bodies (such as cambered ellipses), the use of polar coordinates ($\log r$ vs θ) in Theodorsen's method leads to functions with infinite derivatives or multiple values, making numerical interpolation inaccurate (if not impossible). In contrast, the use of intrinsic coordinates (τ vs s) in James' method always leads to easily interpolated functions (as long as the body has no corners). James' method is thus clearly more suitable for use in transforming complicated shapes. An extreme case that presented no problems for James' method but cannot be transformed using Theodorsen's method is the looped body shown in Fig. 2.

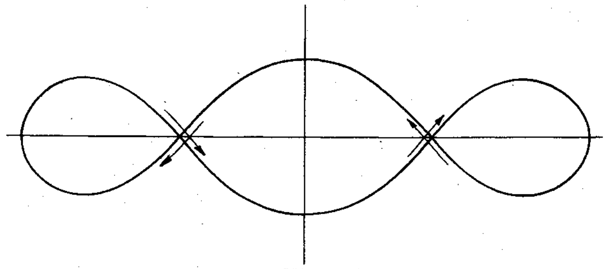

Fig. 2 An extreme case (a looped body) that can be transformed to a unit circle using James' method but not Theodorsen's.

Even excluding those cases for which the use of polar coordinates is inappropriate, James' method still succeeds in many cases where Theodorsen's method fails. To illustrate this, calculations were made, using both methods, on elliptical bodies of various thickness/chord ratios (t/c). Each ellipse was represented by 65 points and was centered on the origin of the coordinate system. For thick bodies ($t/c=0.9$) both methods required about 10 iterations to reduce the residuals to less than 10^{-5}. Both methods required more iterations as the thickness of the bodies was reduced. For bodies of moderate thickness ($t/c=0.4$), Theodorsen's method converged in 25 iterations whereas James' method required 15. Theodorsen's method failed to converge for thinner bodies ($t/c<0.3$). In contrast, James' method was successful in every case tried, down to a minimum thickness/chord ratio of 0.005. As the thickness was reduced, the number of iterations increased, to a maximum of about 25.

A number of possible explanations for the superior convergence characteristics of James' method were considered but ruled out: 1) the initial function estimations used to start the iterations were found to be considerably worse in Theodorsen's method than in James' method, but the use of quite good initial estimates did not significantly improve the convergence characteristics; 2) examination of the functions used in both methods revealed that, in most cases where Theodorsen's method failed, neither method satisfied the sufficient conditions for convergence established by Warschawski.[5]

Detailed examination of the values of all computed quantities finally revealed the mechanism for divergence of the iterations in Theodorsen's method. In general, equally spaced points on the unit circle transform to more closely spaced points in regions with values of $|dz/d\zeta|$ less than unity and to more sparsely spaced points in regions with values greater than unity. The thinner a body gets, the smaller some of the values of $|dz/d\zeta|$ become and the more closely spaced some of the points become. The result of the conjugate function calculation in Theodorsen's method is an approximation to the difference in angular coordinates between corresponding points in the z and ζ planes, from which angular coordinates of points in the z plane are then calculated. In regions where the points are very closely spaced, small errors in the calculated angular coordinates cause a breakdown in the ordering of the points (i.e., point number i has a smaller angular coordinate than point number $i-1$), with the result that subsequent calculations contain larger errors than previous ones and the iterations fail to converge. This can occur even when all estimated values of the series coefficients are in error by less than 0.001! This mechanism for divergence is obviously more likely to occur if larger numbers of points are used to represent the body. Indeed, if the number of points is increased to 257, the iterations fail to converge for the case with $t/c=0.4$, whereas if the number of points is reduced to 17, the iterations do converge for the case with $t/c=0.2$.

This mechanism for failure is not present in James' method. No matter how bad an intermediate result in the iterations may be, the points cannot get out of order. In James' method, the arc lengths of points in the z plane corresponding to equally spaced points in the ζ plane are determined by integrating the calculated values of the modulus of the mapping derivative. Since the integrand is always a positive number, the ordering of the points cannot be changed. As a result, James' method can be used successfully for considerably more complicated cases than Theodorsen's method can.

Acknowledgment

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Flow in Streamwise Corners Having Large Transverse Curvature

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Introduction

A SEMI-INFINITE surface that is everywhere plane except for a single line on which the curvature is infinite is conveniently called a sharp corner. When the line is straight and parallel with an infinite stream in which it is immersed it is called a sharp streamwise corner and is a configuration considered by many workers, e.g., Refs. 1-4. The flow in streamwise corners is interesting in the three-dimensional effects present near to the surface discontinuity, and in its relevance to systems of engineering importance. One particularly interesting feature of the sharp corner is the appearance of the streamwise velocity component as seen in the symmetry plane of the corner. Even in the absence of a freestream pressure gradient this is reminiscent of the profile at the point of separation in two-dimensional flow and consequently suggests a precariously stable condition in the corner flow. The separation-like appearance is directly related to the infinite curvature and the no-slip condition governing the flow at the corner surface. On the other hand, the three-dimensionality in the flow is not conditional on the curvature being infinite but only requires that the curvature should be at least of the same order as the reciprocal of the adjacent two-dimensional boundary-layer thickness δ. It may, therefore, be worthwhile to relax the infinity condition and require only that it be at least $0(\delta^{-1})$; the three-dimensionality will be retained, the unstable appearance of the streamwise velocity profile may be helpfully modified, and the problem to be solved may even be a little more realistic in that practical corners frequently incorporate a small root radius and are seldom absolutely sharp.

These considerations prompted the work presented here, but instead of attempting the "radiused" corner in all its formidable generality, a special case is considered that permits the use of a flow similarity hypothesis and the simplifications arising therefrom. Specifically, the transverse radius of curvature is assumed to be independent of the streamwise coordinates in a suitably transformed coordinate system.

Outline Solution

Consider two imaginary quarter-infinite planes joined at an edge parallel to an infinite uniform stream of speed U and coincident with the positive axis y^1 of a right-handed Cartesian (y^1,y^2,y^3) coordinate system. Let y^1 point downstream and let the planes be symmetrically disposed with respect to the y^2 axis. The analysis will be concerned with the flow along a semi-infinite material surface that everywhere lies in the quarter-infinite planes except for a small region near their joining line in which the surface has a region of finite curvature effecting a smooth transition between its two asymptotes. The arrangement is shown in Fig. 1.

Also shown in Fig. 1 is a curvilinear coordinate system (x^1,x^2,x^3) in which the radiused corner is the surface $x^2=0$. The x^i system is now transformed to a ξ^i system by the relations $\xi^1=x^1$, $\xi^j=(U/\nu x^1)^{1/2}x^j$ $(j=2,3)$; and the y^i to ξ^i transformation is

$$y^1=\xi^1=x^1\equiv x$$

$$y^2=(\nu x/U)^{1/2}\left\{\int_0^\infty(\tan\lambda^*\cos\lambda-\sin\lambda)\,d\xi^3+\xi^2+\int_0^{\xi^3}\sin\lambda\,d\xi^3\right\}\equiv(\nu x/U)^{1/2}I_1$$

$$y^3=(\nu x/U)^{1/2}\int_0^{\xi^3}\cos\lambda\,d\xi^3\equiv(\nu x/U)^{1/2}I_2 \qquad (1)$$

ν is the kinematic viscosity; λ is the angle between the tangent to the surface and the y^3 axis; and it is assumed that $\lambda(x^1,x^3)=\lambda(\xi^3)$, $\lambda(\xi^3)=-\lambda(-\xi^3)$, and that $\lambda\rightarrow\lambda^*$ exponentially fast with increasing ξ^3.

For the given geometry it is expected that a similar solution for the velocity distribution will exist in which the leading terms in expansions of the physical velocity components in the ξ^i direction will be of the form $R_e^{-n_i}v(i)(\xi^2,\xi^3)U$, with

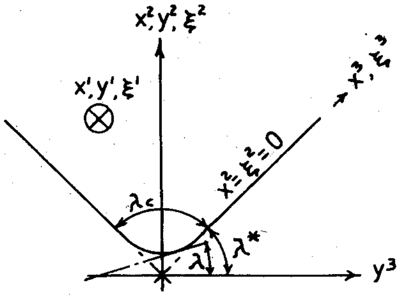

Fig. 1 Curvilinear (x^1,x^2,x^3) and Cartesian (y^1,y^2,y^3) coordinate systems.

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